

# **$G$ -ODOMETERS AND THEIR ALMOST 1-1 EXTENSIONS.**

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**ABSTRACT.** In this paper we recall the concepts of  $G$ -odometer and  $G$ -subodometer for  $G$ -actions, where  $G$  is a discrete finitely generated group, which generalize the notion of odometer in the case  $G = \mathbb{Z}$ . We characterize the  $G$ -regularly recurrent systems as the minimal almost 1-1 extensions of subodometers, from which we deduce that the family of the  $G$ -Toeplitz subshifts coincides with the family of the minimal symbolic almost 1-1 extensions of subodometers.

## 1. INTRODUCTION

It is well known that any continuous dynamical system factorizes onto a minimal equicontinuous dynamical system (see [Au]). For this reason, it is useful to study the minimal equicontinuous factors of a general dynamical system, or conversely, to determine the extensions of a particular minimal equicontinuous system. We place us in this latest problematic. The aim of this paper is to characterize the extensions of a particular type of equicontinuous factor: the  $G$ -odometers, where  $G$  is a discrete finitely generated group, like for example a non Abelian free group. The notion of  $G$ -odometer generalizes the notion of odometer in the case  $G = \mathbb{Z}$ .

An example of extensions of odometers are the Toeplitz flows, which were introduced by Jacobs and Keane in [JK]. Toeplitz flows have been extensively studied in different contexts and they have been used to provide a series of examples with interesting dynamical properties (see for example [Do], [GJ], [Wi]). Markley and Paul characterize them in [MP] as the minimal almost 1-1 extensions of odometers and a proof of this theorem is given in [DL] by Downarowicz and Lacroix.

Following the work developed in [Co] for  $G = \mathbb{Z}^d$ , we prove that for a discrete finitely generated group  $G$ , the  $G$ -Toeplitz systems are the symbolic minimal almost 1-1 extensions of  $G$ -odometers. The main difference that appears with the Abelian case, is the existence of some degenerated systems that we call subodometers.

This paper is organized as follows: in Section 2, we give some basic definitions relevant for the study of topological dynamical systems. We recall also the generalized notions of odometer and subodometer and we identify the set of eigenvalues of these systems. In Section 3, we introduce the notions of regularly recurrent systems and strongly regularly recurrent systems. We characterize them as the minimal almost 1-1 extensions of subodometers and odometers respectively. In the particular case where  $G$  is amenable, we show in Section 4 that the set of invariant probability measures of a  $G$ -regularly recurrent Cantor system can be represented as an inverse limit. In Section 5, in the case when  $G$  is a residually finite group, we introduce a notion of semicycles and we show that an almost 1-1 extension of a  $G$ -subodometer is conjugated to the action of  $G$  on some semicycle. Finally in Section 6, we consider a particular family for a discrete group  $G$ : the  $G$ -Toeplitz arrays, which is a particular family of semicycles when  $G$  is residually finite. We prove, by giving an explicit

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construction, that this family coincides with the family of symbolic almost 1-1 extensions of the  $G$ -subodometers.

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## 2. BASIC DEFINITIONS AND BACKGROUND

In this article, by a *topological dynamical system* we mean a pair  $(X, G)$ , where  $G$  is a topological group which acts, by homeomorphism, on a compact metric space  $(X, d)$ . Given  $g \in G$  and  $x \in X$  we will identify  $g$  with the associated homeomorphism and we denote by  $g.x$  the action of  $g$  on  $x$ . The dynamical system  $(X, G)$  is *free* if  $g.x = x$  for some  $x \in X$  implies  $g = e$ , where  $e$  is the neutral element in  $G$ . For a syndetic subgroup  $\Gamma$  of  $G$ , the  $\Gamma$ -*orbit* of  $x \in X$  is  $O_\Gamma(x) = \{\gamma.x : \gamma \in \Gamma\}$  and the  $\Gamma$ -*system associated* to  $x$  is  $(\Omega_\Gamma(x), \Gamma)$ , where  $\Omega_\Gamma(x)$  is the closure of  $O_\Gamma(x)$  and the action of  $\Gamma$  on  $\Omega_\Gamma(x)$  is the restriction to  $\Gamma$  and  $\Omega_\Gamma(x)$  of the action of  $G$  on  $X$ . The set of *return times* of  $x \in X$  to  $A \subseteq X$  is  $T_A(x) = \{g \in G : g.x \in A\}$ . The topological dynamical system  $(X, G)$  is *minimal* if the orbit of any  $x \in X$  is dense in  $X$ , and it is said to be *equicontinuous* if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $x, y \in X$  satisfy  $d(x, y) < \delta$  then  $d(g.x, g.y) < \varepsilon$  for all  $g \in G$ . We say that  $(X, G)$  is an *extension* of  $(Y, G)$ , or that  $(Y, G)$  is a *factor* of  $(X, G)$ , if there exists a continuous surjection  $\pi : X \rightarrow Y$  such that  $\pi$  preserves the action. We call  $\pi$  a *factor map*. When the factor map is bijective, we say that  $(X, G)$  and  $(Y, G)$  are *conjugate*. The factor map  $\pi$  is an *almost 1-1 factor map* and  $(X, G)$  is an *almost 1-1 extension* of  $(Y, G)$  by  $\pi$  if the set of points having one pre-image is residual (contains a dense  $G_\delta$  set) in  $Y$ . In the minimal case it is equivalent to the existence of a point with one pre-image.

The set  $\mathcal{M}_G(X)$  of *invariant probability measures* of  $X$  is the set of probability measures  $\mu$  defined on  $\mathcal{B}(X)$ , the Borel  $\sigma$ -algebra of  $X$ , such that  $\mu(g.B) = \mu(B)$  for all  $g \in G$  and  $B \in \mathcal{B}(X)$ .

**2.1.  $G$ -odometers and  $G$ -subodometers.** In all the following, we will denote by  $G$  a discrete group generated by a finite family and by  $e$  its neutral element.

**Definition 1.** A discrete finitely generated group  $G$  is called *residually finite* if and only if there exists a sequence  $\Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_n \supset \dots$  of subgroups  $\Gamma_n$  with finite index in  $G$  such that:

$$\bigcap_n \Gamma_n = \{e\}.$$

A trivial example of a residually finite subgroup is the group of integers  $\mathbb{Z}$ , for example by taking the groups  $\Gamma_n = n!\mathbb{Z}$ . Less trivial examples are given by the fundamental groups of connected oriented compact graph. When  $\pi : S_2 \rightarrow S_1$  is a finite covering of an oriented compact connected graph  $S_2$  onto a compact graph  $S_1$ , the application  $\pi$  induces an homomorphism  $\pi_*$  from the fundamental group of  $S_2$  to the fundamental group of  $S_1$ . The image of the morphism  $\pi_*$  is a subgroup of the fundamental group of  $S_1$ . The index of this subgroup is then the number of pre-image of one point for the map  $\pi$ . Let us denote by  $\widetilde{S}_1$  the universal cover of  $S_1$ . Consider a sequence  $(S_n, \pi_n)_n$  of finite covering  $\pi_n : S_{n+1} \rightarrow S_n$  of compact conneced and oriented graph  $S_n$  such that for each  $n$  the injectivity radius of  $\widetilde{S}_1$  onto  $S_n$  goes to infinity when  $n$  goes to infinity. The sequence of fundamental groups of graphs  $S_n$  satisfies then the condition of the Definition 1. More generally, we have the following result of Mal'cev [Ma]:

**Theorem 1.** [Ma] *For any integer  $n$  and any field  $\mathbb{K}$  with characteristic null, every subgroup finitely generated of the group of invertible matrices  $GL(n, \mathbb{K})$  is a residually finite groups.*

In particular, the free groups  $\mathbb{F}_n$  with  $n$  generators, the groups of surfaces and the braids group  $B_n$  generated by  $n$  elements are residually finite groups.

Let us denote, for a subgroup  $H$  of  $G$ , by  $G/H$  the set of right class of  $H$  in  $G$ . It is important to note that  $G$  acts on  $G/H$  by left multiplication on the  $H$ -class. Now we will prove the useful following lemma:

**Lemma 1.** *Let  $G$  be a group. If  $H$  is a subgroup of  $G$  with index in  $G$  equal to  $n$  (i.e. the cardinal of the quotient space  $G/H$  is  $n$ ) then there exists a normal subgroup  $K$  of  $G$  included in  $H$  such that  $G/K$  divide  $n!$ .*

*Proof.* The group  $G$  acts on  $G/H$  by left multiplication. This action defines an homomorphism  $\rho$  from  $G$  to the permutation group of  $n$  elements. The kernel of this application is a normal subgroup of  $G$  included in  $H$  and its index in  $G$  divides the cardinal of permutations of  $n$  elements.  $\square$

As a corollary,  $G$  is a residually finite group if and only if there exists a sequence  $H_1 \supset \dots \supset H_n \supset \dots$  of normal subgroups of  $G$  with finite index in  $G$  such that  $\bigcap_n H_n = \{e\}$ . Remark that, up to consider a quotient space, all finitely generated groups are residually finites.

Let us consider a discrete group  $G$  generated by a finite family and a decreasing sequence (for the inclusion)  $(\Gamma_i)_{i \geq 0} \subseteq G$  of subgroups with finite index in  $G$  (we do not ask  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$ ) and let  $\pi_i : G/\Gamma_{i+1} \rightarrow G/\Gamma_i$  be the function induced by the inclusion  $\Gamma_{i+1} \subset \Gamma_i$ ,  $i \geq 0$ . Consider the inverse limit

$$\overleftarrow{G} = \varprojlim_i (G/\Gamma_i, \pi_i).$$

More precisely,  $\overleftarrow{G}$  is defined as the subset of the product  $\prod_{i \geq 0} G/\Gamma_i$  consisting of the elements  $\mathbf{g} = (g_i)_{i \geq 0}$  such that  $\pi_i(g_{i+1}) = g_i$  for all  $i \geq 0$ .

Every  $G/\Gamma_i$  is endowed with the discrete topology and  $\prod_{i \geq 0} G/\Gamma_i$  with the product topology. Thus  $\overleftarrow{G}$  is a compact metrizable space whose topology is spanned by the cylinder sets

$$[i; a] = \{\mathbf{g} \in \overleftarrow{G} : g_i = a\}, \text{ with } a \in G/\Gamma_i \text{ and } i \geq 0.$$

The space  $\overleftarrow{G}$  is a totally disconnected, it is a Cantor set when  $G/\bigcap_{i \geq 0} \Gamma_i$  is infinite and a finite set when  $G/\bigcap_{i \geq 0} \Gamma_i$  is finite.

The group  $G$  acts continuously on  $\overleftarrow{G}$  by left multiplication, namely for  $\mathbf{g} = (g_i)_i \in \overleftarrow{G}$  and  $h \in G$ ,

$$h.\mathbf{g} = (h.g_i)_i,$$

where  $h.g_i$  denotes the action on  $G/\Gamma_i$  given by  $h.g_i = hg\Gamma_i$ , for every  $h \in G$  and  $g \in G$ . Since for all  $h \in G$  and for all cylinders  $[i; a]$  we have

$$h.([i; a]) \subseteq [i; h.g_i],$$

the topological dynamical system  $(\overleftarrow{G}, G)$  is equicontinuous. Moreover, every orbit for this action is dense, then  $(\overleftarrow{G}, G)$  is a minimal equicontinuous system.

**Definition 2.** *We call  $(\overleftarrow{G}, G)$  a  $G$ -subodometer system\* or simply a subodometer. If in addition, every  $\Gamma_i$  is normal, we say that  $(\overleftarrow{G}, G)$  is a  $G$ -odometer system or simply an odometer.*

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\*Note that this definition is not a profinite completion of the group  $G$  because here, we consider only a sequence of decreasing subgroups.

It is straightforward to show that for a point  $\mathbf{g} = (g_i)_i$  of a subodometer  $\overleftarrow{G}$ , its stabilizer for the  $G$ -action is the group  $\bigcap_i \tilde{g}_i \Gamma_i \tilde{g}_i^{-1}$ , where  $\tilde{g}_i$  is a representing element of the class  $g_i \in G/\Gamma_i$  in  $G$ , for  $i \geq 0$ . Hence, when  $G$  is a residually finite group and  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$ , for  $\mathbf{e}$  the element  $(e_i)_i$  of the  $G$ -subodometer  $\overleftarrow{G}$ , where  $e_i$  is the projection of the neutral element of  $G$  on  $G/\Gamma_i$ , the stabilizer of  $\mathbf{e}$  is trivial. This does not mean necessarily that the action of  $G$  on  $\overleftarrow{G}$  is free. If furthermore, all the groups  $\Gamma_i$  are normal subgroups of  $G$ , then the stabilizer of every point of a  $G$ -odometer is trivial and the action of  $G$  is free. For this reason we call, when  $G$  is residually finite and  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$ , the  $G$ -odometer  $\lim_{\leftarrow i} (G/\Gamma_i, \pi_i)$  a *free  $G$ -odometer*.

If  $(\overleftarrow{G}, G)$  is an odometer then the set  $\overleftarrow{G}$  is a group equipped with the multiplication defined by

$$\mathbf{g} \cdot \mathbf{h} = (g_i \cdot_i h_i)_{i \geq 0},$$

where  $\cdot_i$  denotes the multiplication operation induced on  $G/\Gamma_i$  by the multiplication on  $G$ .

Remark that for a free odometer  $(\overleftarrow{G}, G)$ , the group  $G$  is then a dense subgroup of  $\overleftarrow{G}$ .

Notice that for all  $\mathbf{g}$  in a cylinder set  $[i; a]$  of an odometer  $\overleftarrow{G} = \lim_{\leftarrow i} (G/H_i, \pi_i)$ , the set of return times of  $\mathbf{g}$  to  $[i; a]$  is  $H_i$ . Through this paper we will use this property and we will identify  $\overleftarrow{G}$  with  $(\overleftarrow{G}, G)$ .

**Lemma 2.** *Let  $\overleftarrow{G}_j = \lim_{\leftarrow i} (G/H_i^j, \pi_i)$  be two subodometers ( $j = 1, 2$ ). Let  $\mathbf{e}_j$  ( $j = 1, 2$ ) be the element  $(e_i^j)_i \in \overleftarrow{G}_j$  where  $e_i^j$  denotes the class of the neutral element  $e \in G$  in  $G/H_i^j$ .*

*There is a factor map  $\pi : (\overleftarrow{G}_1, G) \rightarrow (\overleftarrow{G}_2, G)$  such that  $\pi(\mathbf{e}_1) = \mathbf{e}_2$  if and only if for every  $H_i^2$  there exists some  $H_k^1$  such that  $H_k^1 \subseteq H_i^2$ .*

*Proof.* If  $\pi : \overleftarrow{G}_1 \rightarrow \overleftarrow{G}_2$  is a factor map then by continuity, given  $i \geq 0$  and  $e_i^2$  in  $G/H_i^2$ , there exists  $k \geq 0$  such that  $[k; e_k^1] \subseteq \pi^{-1}[i; e_i^2]$ . Let  $v \in H_k^1$ , we have that  $v \cdot \mathbf{g} \in [k; e_k^1]$  for all  $\mathbf{g} \in [k; e_k^1]$ , which implies that

$$\pi(v \cdot \mathbf{g}) = v \cdot \pi(\mathbf{g}) \in [i; e_i^2].$$

Since  $\pi(\mathbf{g}) \in [i; e_i^2]$  and  $T_{[i; e_i^2]}(\pi(\mathbf{g})) = H_i^2$ , we get  $v \in H_i^2$ .

Suppose that for every  $i \geq 0$  there exists  $H_{n_i}^1 \subseteq H_i^2$ . Since the sequences  $(H_i^j)_{i \geq 0}$ ,  $j = 1, 2$ , are decreasing, we can take  $n_i \leq n_{i+1}$  for all  $i \geq 0$ . The function  $\pi : \overleftarrow{G}_1 \rightarrow \overleftarrow{G}_2$  defined by  $\pi((g_i)_{i \geq 0}) = (j_{n_i}(g_{n_i}))_{i \geq 0}$  where  $j_{n_i} : G/H_{n_i}^1 \rightarrow G/H_i^2$  is the function induced by the inclusion  $H_{n_i}^1 \subseteq H_i^2$ , is a factor map.  $\square$

By a straightforward application of the former lemma and Lemma 1, we get

**Proposition 1.** *If  $(\lim_{\leftarrow i} (G/\Gamma_i, \pi_i), G)$  is a  $G$ -subodometer, then there exists a  $G$ -odometer which is an extension of this subodometer.*

**Proposition 2.** *For  $\overleftarrow{G}$  a  $G$ -odometer and  $(X, G)$  a dynamical system, if there exists a factor map from  $\overleftarrow{G}$  onto  $X$ , then there exists a closed subgroup  $H$  of  $\overleftarrow{G}$  such that the dynamical system  $(\overleftarrow{G}/H, G)$  is conjugated to  $(X, G)$ .*

In particular this proposition says that a subodometer is conjugate to the quotient of an odometer by a closed subgroup.

*Proof.* Let us denote by  $p$  the factor map  $\overleftarrow{G} \rightarrow X$ ,  $\mathbf{e}$  the neutral element of  $\overleftarrow{G}$  and for  $\mathbf{g}$  an element of  $\overleftarrow{G}$ , we denote by  $(g_i)_i$  a sequence of  $G \subset \overleftarrow{G}$  that converges to  $\mathbf{g}$ . Let  $H$  be the closed subset  $p^{-1}(p(\mathbf{e}))$  of  $\overleftarrow{G}$ . For  $\mathbf{g}$  and  $\mathbf{h}$  in  $H$ , we have:

$$p(\mathbf{hg}) = \lim_i p(h_i g_i) = \lim_i h_i \cdot p(g_i) = \lim_i h_i \cdot p(\mathbf{e}) = \lim_i p(h_i) = p(\mathbf{e}).$$

With the same technic we get:

$$p((\mathbf{g})^{-1}) = \lim_i p(g_i^{-1} \mathbf{e}) = \lim_i g_i^{-1} \cdot p(\mathbf{e}) = \lim_i g_i^{-1} \cdot p(g_i) = p(\mathbf{e}).$$

So  $\mathbf{gh}$  and  $\mathbf{g}^{-1}$  belong to  $H$ , and  $H$  is a group.

Now let us see that  $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$  for any  $\mathbf{g} \in \overleftarrow{G}$ . Let  $\mathbf{h}$  be in  $H$ , we have:

$$p(\mathbf{gh}) = \lim_i p(g_i h_i) = \lim_i g_i \cdot p(h_i) = \lim_i g_i \cdot p(\mathbf{e}) = p(\mathbf{g}).$$

Then  $\mathbf{g}H \subset p^{-1}(p(\mathbf{g}))$ .

Let  $\mathbf{h} \in \overleftarrow{G}$  be such that  $p(\mathbf{h}) = p(\mathbf{g})$ . Then  $\lim_i p(g_i) = \lim_i p(h_i)$  and  $p(\mathbf{e}) = \lim_i g_i^{-1} \cdot h_i \cdot p(\mathbf{e}) = p(\mathbf{g}^{-1}\mathbf{h})$ . So  $\mathbf{g}^{-1}\mathbf{h}$  belongs to  $H$  and  $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$ . Therefore, the map  $p$  factorizes onto a homeomorphism from  $\overleftarrow{G}/H$  to  $X$ .  $\square$

**2.2. Eigenvalues of odometers and subodometers.** Let  $(X, \mu, G)$  be a measure-theoretic dynamical system with a left action of  $G$ . A character  $\chi$  is a homomorphism from  $G$  to the group  $\mathbb{S}^1$ , the set of complex numbers with module 1. Since the group  $G$  is equipped with the discrete topology, every character is a continuous map.

A character is an *eigenvalue* of  $X$  if there exists  $f \in L^2_\mu(X) \setminus \{0\}$  such that  $f(g.x) = \chi(g)f(x)$  for all  $x \in X$  and  $g \in G$ . We call  $f$  an *eigenfunction* associated to  $\chi$ . We say that an eigenvalue is a *continuous eigenvalue* if it has an associated continuous eigenfunction.

Since a  $G$ -odometer  $\overleftarrow{G}$  is a compact group, the normalized Haar measure left invariant  $\lambda$  of  $\overleftarrow{G}$  is the only invariant probability measure of  $\overleftarrow{G}$  for the action of  $G$ . A  $G$ -subodometer is a factor of a  $G$ -odometer (Proposition 1), therefore, by this factorization, the subodometer inherits also of an invariant measure for the  $G$ -action. Since any factor map extends to a continuous affine onto map between the set of invariant probability measures [DGS], we conclude that a subodometer is uniquely ergodic. Thus when we speak about a subodometer  $\overleftarrow{G}$  as a measure-theoretic dynamical system, we mean  $\overleftarrow{G}$  equipped with the only invariant probability measure  $\lambda$  for the action of  $G$ .

**Proposition 3.** *Let  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  be a subodometer. The set of eigenvalues of  $\overleftarrow{G}$  is given by  $E_G = \bigcup_{n \geq 0} \{\text{character } \chi : G \rightarrow \mathbb{S}^1, \chi(\gamma) = 1 \text{ for } \gamma \in \Gamma_n\}$ . Moreover, every eigenvalue of  $\overleftarrow{G}$  is a continuous eigenvalue.*

*Proof.* For  $n \geq 0$  we call  $C_n = [n; e]$ . Since  $v, w \in G$  satisfy  $v.C_n = w.C_n$  if and only if  $w$  and  $v$  belong to the same class in  $G/\Gamma_n$ , it makes sense to write  $v.C_n$  for  $v \in G/\Gamma_n$ . Notice that the collection  $\mathcal{P}_n = \{v.C_n : v \in G/\Gamma_n\}$  is a clopen partition of  $G$ .

Let  $\chi \in E_G$  and let  $n \geq 0$  be such that  $\chi(\gamma) = 1$  for all  $\gamma \in \Gamma_n$ . This means that  $\chi$  is constant on each class of  $G/\Gamma_n$ , which implies that  $f = \sum_{v \in G/\Gamma_n} \chi(v) 1_{v.C_n}$  is a well defined continuous function that verifies  $f(h.\mathbf{g}) = \chi(h)f(\mathbf{g})$  for all  $\mathbf{g} \in \overleftarrow{G}$  and  $h \in G$ .

Let  $\chi$  be an eigenvalue of  $\overleftarrow{G}$  and let  $f \in L^2_\lambda(\overleftarrow{G}) \setminus \{0\}$  be an associated eigenfunction. For  $g \in G$  we have that

$$\chi(g) \left( \int_{C_n} f d\lambda \right) = \int_{g.C_n} f d\lambda.$$

Since  $C_n = \gamma.C_n$  for all  $\gamma \in \Gamma_n$ , it holds that

$$(1) \quad \chi(g) \left( \int_{C_n} f d\lambda \right) = \int_{C_n} f d\lambda \quad \text{for all } g \in \Gamma_n.$$

Observe that

$$\mathbb{E}(f|\mathcal{P}_n) = \sum_{g \in K_n} \frac{\chi(g)}{\lambda(C_n)} \left( \int_{C_n} f d\lambda \right) 1_{g.C_n},$$

for a finite set  $K_n \subset G$  containing at least one element of each class of  $G/\Gamma_n$ . Since  $\mathcal{B}(\mathcal{P}_n) \uparrow \mathcal{B}(\overleftarrow{G})$ , by the increasing Martingale theorem, we have that  $\mathbb{E}(f|\mathcal{P}_n)$  converges to  $f$  in  $L^2_\lambda(\overleftarrow{G})$ . Because  $f \neq 0$ , this implies there exists  $m \geq 0$  such that  $\int_{C_m} f d\lambda \neq 0$  and, by (1), we conclude that  $\chi(\gamma) = 1$  for all  $\gamma \in \Gamma_m$ , which means that  $\chi \in E_G$ .  $\square$

### 3. CHARACTERIZATION OF MINIMAL ALMOST 1-1 EXTENSIONS OF ODOMETERS

Let  $(X, G)$  and  $(Y, G)$  be two topological dynamical systems.  $(Y, G)$  is said to be the *maximal equicontinuous factor* of  $(X, G)$  if it is an equicontinuous factor of  $(X, G)$  such that for any other equicontinuous factor  $(Y', G)$  of  $(X, G)$  there exists a factor map  $\pi : Y \rightarrow Y'$  that satisfies  $\pi \circ f = f'$ , with  $f : X \rightarrow Y$  and  $f' : X \rightarrow Y'$  factor maps.

It is well known that every topological dynamical system has a maximal equicontinuous factor and if  $(X, G)$  is a minimal almost 1-1 extension of a minimal equicontinuous system  $(Y, G)$ , then  $(Y, G)$  is the maximal equicontinuous factor of  $(X, G)$  (for more details see [Au]).

**3.1. Regularly recurrent systems.** A subset  $S$  of  $G$  is said to be *syndetic* if there exists a compact subset  $K$  of  $G$  such that  $G = K.S = \{k.s : s \in S, k \in K\}$ . Because we consider  $G$  a discrete group, a subset  $S$  of  $G$  is syndetic if and only there exists a finite subset  $K$  of  $G$  such that  $G = K.S$ . It is important to note that a subgroup  $\Gamma$  of  $G$  is syndetic if and only if  $G/\Gamma$  is finite.

Let  $(X, G)$  be a topological dynamical system and let  $x \in X$ . The point  $x$  is *uniformly recurrent* if for every open neighborhood  $V$  of  $x$  the set  $T_V(x)$  is syndetic. It is well known that  $(\Omega_G(x), G)$  is minimal if and only if  $x$  is uniformly recurrent.

A point  $x \in X$  is *regularly recurrent* if for every open neighborhood  $V$  of  $x$  there is a syndetic subgroup  $\Gamma$  of  $G$  such that  $\Gamma \subseteq T_V(x)$ . We say that a system is *regularly recurrent* if it is the orbit closure of a regularly recurrent point.

Similarly, we say that a point  $x \in X$  is *strongly regularly recurrent* if for every open neighborhood  $V$  of  $x$  there is a clopen subset  $W \subset V$ , neighborhood of  $x$ , such that  $T_W(x)$  is a syndetic normal subgroup of  $G$ . We say that a system is *strongly regularly recurrent* if it is the orbit closure of a strongly regularly recurrent point. Obviously, a strongly regularly recurrent point is a regularly recurrent point. Regularly recurrent systems are minimal.

The subodometers (resp. odometers) are examples of (resp. strongly) regularly recurrent systems. Moreover, every point in a subodometer (resp. odometer)  $\overleftarrow{G}$  is regularly recurrent (resp. strongly regularly recurrent).

In this section, we will show that (resp. strongly) recurrent systems are exactly the minimal almost 1-1 extensions of the subodometers (resp. odometers). From that we will conclude that a group  $G$  admits an action that is strongly regularly recurrent and free if and only if  $G$  is residually finite.

**Lemma 3.** *Let  $(X, G)$  be a minimal topological dynamical system and let  $x \in X$ . If  $\Gamma \subseteq G$  is a syndetic subgroup of  $G$  then  $(\Omega_\Gamma(x), \Gamma)$  is minimal.*

*Proof.* Let  $H$  be a normal subgroup of  $G$  included in  $\Gamma$  (Lemma 1). The group  $G$  acts by the natural product action on the compact spaces  $X \times G/H$  and  $X \times G/\Gamma$ . Pick a minimal set  $M$  in  $X \times G/H$ . This set projects onto a minimal subset of  $X$  hence onto  $X$ . Thus for every  $x \in X$  there exists a point  $(x, a) \in M$  and this point is uniformly recurrent. The right multiplication by  $a^{-1}$  on the second axis is a conjugacy that sends the minimal set  $M$  onto a minimal set  $M'$  that contains  $(x, e)$ . This set projects onto a minimal set of  $X \times G/\Gamma$  that contains the point  $(x, [e])$  where  $[e]$  denotes the  $\Gamma$ -class of the neutral element  $e$ . This implies that for any neighborhood  $V \subseteq X$  of  $x$ , the set  $\{g : g.x \in V, g \in \Gamma\}$  is syndetic.  $\square$

**Lemma 4.** *Let  $(X, G)$  be a topological dynamical system and let  $x \in X$  be a (resp. strongly) regularly recurrent point. For every closed neighborhood  $V$  of  $x$  there exists a (resp. normal) syndetic subgroup  $\Gamma$  of  $G$  such that  $\Gamma \subseteq T_V(x)$  and  $\{w.\Omega_\Gamma(x)\}_{w \in G/\Gamma}$  is a clopen partition of  $X$ .*

*Proof.* Let  $V \subseteq X$  be a closed neighborhood of a point  $x$  regularly recurrent and let  $\Gamma' \subseteq G$  be a subgroup with finite index such that  $\Gamma' \subseteq T_V(x)$ . Let us consider the normal subgroup  $H \subset \Gamma'$  given by Lemma 1. By Lemma 3, the set  $\Omega_H(x)$  is a closed set minimal and invariant for the  $H$ -action. Since  $H$  is normal, for any  $g \in G$ , the set  $g.\Omega_H(x)$ , which equals  $\Omega_H(g.x)$ , is also closed, invariant and minimal for the  $H$ -action. Therefore if  $w.\Omega_H(x) \cap u.\Omega_H(x) \neq \emptyset$  for  $u, w \in G$ , we have  $w.\Omega_H(x) = u.\Omega_H(x)$ .

Furthermore, if  $u$  and  $w \in G$  are in the same  $H$ -class, then we have also  $w.\Omega_H(x) = u.\Omega_H(x)$ . Since  $H$  is syndetic and the  $G$ -orbit of  $x$  is dense, we have  $X = \bigsqcup_{u \in K} u.\Omega_H(x)$ , for some finite family  $K$  of  $G$ .

Let  $\Gamma$  be the group

$$\Gamma = \{g \in G : g.\Omega_H(x) = \Omega_H(x)\}.$$

We have  $H \subset \Gamma$ , so  $\Gamma$  is syndetic. Since  $\Omega_\Gamma(x) = \Omega_H(x)$ , we have  $\Gamma \subset T_V(x)$  and for any  $g \in G$   $g.\Omega_\Gamma(x)$  and  $\Omega_\Gamma(x)$  are disjoint or equal because they are minimal closed  $H$ -invariant sets. Thus we get :

- (1)  $g.\Omega_\Gamma(x) = g'.\Omega_\Gamma(x)$  if and only if  $g \in g'\Gamma$ .
- (2)  $T_{g.\Omega_\Gamma(x)}(y) = g\Gamma g^{-1}$  for every  $y \in g.\Omega_\Gamma(x)$ .

It holds that for  $w \in G/\Gamma$ ,  $w.\Omega_\Gamma(x)$  is well defined and  $\{w.\Omega_\Gamma(x)\}_{w \in G/\Gamma}$  is a clopen partition of  $X$ .

When  $x$  is a strongly regularly recurrent point of  $X$ , we do exactly the same proof with  $H$  being the normal subgroup  $T_W(x)$  given by a clopen neighborhood  $W \subset V$  of  $x$ . Thanks this strong property, we have that the group  $\Gamma$  is actually the group  $H$  and thus  $\Gamma$  is a normal subgroup of  $G$ .  $\square$

**Corollary 1.** *Let  $(X, G)$  be a topological dynamical system and let  $x \in X$ . The point  $x$  is (resp. strongly) regularly recurrent if and only if there exists  $(C_i)_{i \geq 0}$ , a fundamental system of clopen neighborhoods of  $x$  ( $\cap_i C_i = \{x\}$ ), such that for all  $y \in C_i$  the set of return times of  $y$  to  $C_i$  is a syndetic (resp. normal) subgroup  $\Gamma_i$  of  $G$ , for every  $i \geq 0$ .*

*Proof.* If  $x \in X$  has a fundamental system of neighborhoods as written above, it is a (resp. strongly) regularly recurrent point.

The sequences  $(C_i)_i$  and  $(\Gamma_i)_i$  are defined by induction. If  $x$  is a (resp. strongly) regularly recurrent point, let  $C_1$  be the space  $X$  and  $\Gamma_1$  be the group  $G$ .

So, given  $C_n$  and  $\Gamma_n$ , we take an open neighborhood  $V_{n+1}$  of  $x$ , whose the closure is strictly included in  $C_n$ . By Lemma 4, we obtain a syndetic (resp. normal) group  $\Gamma_{n+1}$  with  $\Gamma_{n+1} \subseteq$

$T_{\overline{V}_{n+1}}(x)$  and  $\{w(\Omega_{\Gamma_{n+1}}(x))\}_{w \in G/\Gamma_{n+1}}$  is a clopen partition of  $X$ . Clearly, we have  $\Gamma_{n+1} \subset \Gamma_n$ . We set  $C_{n+1} = \Omega_{\Gamma_{n+1}}(x)$  which is a clopen set with  $T_{C_{n+1}}(y) = \Gamma_{n+1}$  for all  $y \in \Gamma_{n+1}$ . Since  $\lim_{i \rightarrow \infty} \text{diam}(V_n) = 0$ , we obtain that  $(C_i)_{i \geq 0}$  is a fundamental system of clopen neighborhoods of  $x$ .  $\square$

**Theorem 2.** *A minimal topological dynamical system  $(X, G)$  is an almost 1-1 extension of a subodometer (resp. odometer)  $\overleftarrow{G}$  by  $\pi$  if and only if  $(X, G)$  is a (resp. strongly) regularly recurrent system. Moreover, the set of (resp. strongly) regularly recurrent points of  $X$  is exactly the pre-image of the set of points in  $G$  which have only one pre-image by  $\pi$ .*

*Proof.* Let  $(X, G)$  be a minimal 1-1 extension of an subodometer  $\overleftarrow{G} = \varprojlim_i (G/\Gamma_i, \pi_i)$  (resp. odometer). Let  $\pi : X \rightarrow \overleftarrow{G}$  be the almost 1-1 factor map and let  $x \in X$  be such that  $\{x\} = \pi^{-1}(\{\pi(x)\})$ . Since  $\pi$  is continuous, if  $\pi(x) = (a_i)_{i \geq 0} \in \overleftarrow{G}$  then  $(\pi^{-1}([i; a_i]))_i$  is a decreasing sequence of clopen neighborhoods of  $x$  that satisfies

$$\bigcap_{i \geq 0} \pi^{-1}([i; a_i]) = \{x\}.$$

We know that for every  $\mathbf{g} \in [i; a_i]$ , the set  $T_{[i; a_i]}(\mathbf{g})$  is a group conjugated to  $\Gamma_i$ , therefore for all  $y$  in  $\pi^{-1}([i; a_i])$ , we have  $T_{\pi^{-1}([i; a_i])}(y)$  is a group conjugated to  $\Gamma_i$ . So, by Corollary 1 we conclude that  $x$  is a (resp. strongly) regularly recurrent point of  $X$ .

Let  $X$  be a (resp. strongly) regularly recurrent system and let  $x \in X$  be a (resp. strongly) regularly recurrent point with a trivial stabilizer. By Corollary 1 there exists a decreasing sequence  $(C_i)_{i \geq 0}$  of clopen neighborhoods of  $x$  such that  $\bigcap_{i \geq 0} C_i = \{x\}$ , and there is a syndetic (resp. normal) subgroup  $\Gamma_i$  such that  $T_{C_i}(y) = \Gamma_i$  for all  $y \in C_i$ ,  $i \geq 0$ . Since  $C_{i+1} \subseteq C_i$ , we have that  $\Gamma_{i+1} \subseteq \Gamma_i$ ,  $i \geq 0$ . So, we can define the subodometer (resp. odometer)  $\overleftarrow{G} = \varprojlim_i (G/\Gamma_i, \pi_i)$ . We define  $\pi : X \rightarrow \overleftarrow{G}$  by  $\pi = (f_i)_{i \geq 0}$  where  $f_i$  is the continuous map  $f_i : X \rightarrow G/\Gamma_i$  given by  $f_i(y) = [z]$ , where  $[z]$  denotes the  $\Gamma_i$ -class of  $z \in G$ , if and only if  $y \in z.C_i$  for  $y \in X$ ,  $z \in G$  and  $i \geq 0$ . The function  $\pi$  is a factor map, and, since  $\bigcap_{i \geq 0} C_i = \{x\}$ , we have that  $\pi^{-1}\{\mathbf{e}\} = \{x\}$ . So,  $\pi$  is an almost 1-1 extension.

If  $\pi' : X \rightarrow \overleftarrow{G}'$  is another almost 1-1 factor map and  $\overleftarrow{G}'$  an subodometer (resp. odometer),  $\overleftarrow{G}$  and  $\overleftarrow{G}'$  are the maximal equicontinuous factor of  $(X, G)$ , therefore, they are conjugate. Thus there exists a factor map  $\pi'' : \overleftarrow{G}' \rightarrow \overleftarrow{G}$  such that  $\pi'' \circ \pi' = \pi$ , which implies that  $\pi'^{-1}\{x\} = \pi^{-1}\{\pi''(x)\}$  for any  $x$  of  $\overleftarrow{G}'$ . We conclude that the set of (resp. strongly) regularly recurrent points is exactly the pre-image of the points in  $G$  which have only one pre-image.  $\square$

By a straightforward application of Theorem 2 we get the following corollaries.

**Corollary 2.** *Every point of a (resp. strongly) regularly recurrent system  $(X, G)$  is (resp. strongly) regularly recurrent if and only if  $(X, G)$  is conjugate to a (resp. odometer) subodometer.*

**Corollary 3.** *A discrete group finitely generated  $G$  admits a strongly regularly recurrent free action on a compact metric space if and only if  $G$  is residually finite.*

**Corollary 4.** *Let  $(X, G)$  be a regularly recurrent system and let  $\overleftarrow{G}$  be its maximal equicontinuous factor. The set of continuous eigenvalues of  $X$  is  $E_G$ .*



*Proof.* It is clear that  $E_G$  is contained in the set of continuous eigenvalues of  $X$ . Conversely, if  $\chi$  is a continuous eigenvalue of  $X$  we can take  $f : X \rightarrow \mathbb{S}^1$  an associated continuous eigenfunction which is a factor map between  $(X, G)$  and the dynamical system  $(f(X), G)$ , where the action of  $g \in G$  on  $\exp(2i\pi x) \in f(X)$  is given by  $g \cdot \exp(2i\pi x) = \chi(g) \exp(2i\pi x)$ , which is an isometry. Thus the system  $(f(X), G)$  is equicontinuous and therefore there exists a factor map  $\pi : \overleftarrow{G} \rightarrow f(X)$ . Since  $\pi$  is an eigenfunction associated to  $\chi$  we conclude that  $\chi \in E_G$ .  $\square$

#### 4. REGULARLY RECURRENT CANTOR SYSTEMS WITH $G$ AMENABLE.

We say that a topological dynamical system  $(X, G)$  is a (*resp. strongly*) *regularly recurrent Cantor system* if it is (*resp. strongly*) regularly recurrent and  $X$  is a Cantor set. In this section we suppose that  $(X, G)$  is a regularly recurrent Cantor system.

**Proposition 4.** *Let  $(X, G)$  be a regularly recurrent Cantor system. There exists a sequence*

$$\mathcal{P}_n = \{w.C_{n,k} : w \in D_n, 1 \leq k \leq k_n\},$$

*of finite clopen partitions of  $X$  satisfying, for every  $n \geq 0$ , the following:*

- (1)  $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} C_{n,k} \subset X$ .
- (2) *There exists a syndetic subgroup  $\Gamma_n$  of  $G$  such that  $D_n$  is a subset of  $G$  containing exactly one representing element of each class in  $G/\Gamma_n$  and such that  $T_{C_n}(x) = \Gamma_n$ , for all  $x \in C_n$ .*
- (3)  $\mathcal{P}_{n+1}$  is finer than  $\mathcal{P}_n$ .
- (4) *The collection of set  $(P_n)_{n \geq 0}$  spans the topology of  $X$ .*

*Proof.* The idea of the proof is the same used in [HPS] and [Pu] to show that any minimal Cantor  $\mathbb{Z}$ -system has a nested sequence of clopen Kakutani-Rohlin partitions.

We recall the algorithm introduced in [Pu] to generate a Kakutani-Rohlin partition finer than another one. Let  $\mathcal{R}$  be a finite clopen partition of  $X$ . Suppose that

$$\mathcal{Q} = \{w.C_j : w \in D, 1 \leq j \leq k\},$$

is another clopen partition of  $X$  for which  $k < \infty$ , there exists a syndetic subgroup  $\Gamma$  of  $G$  such that  $D = \{w_1, \dots, w_l\}$  is a subset of  $G$  containing exactly one representing element of each class in  $G/\Gamma$ , and the set of return times of any point in  $C = \bigcup_{j=1}^k C_j$  to  $C$  is equal to  $\Gamma$ . The next algorithm produce a partition  $\mathcal{R} \wedge \mathcal{Q} = \{w.B_j : w \in D, 1 \leq j \leq d\}$  verifying

- $\mathcal{R} \wedge \mathcal{Q}$  is finer than  $\mathcal{R}$  and  $\mathcal{Q}$ .
- $C = \bigcup_{j=1}^d B_j$

Step 1: let  $1 \leq j \leq k$ . Consider  $A_{1,j,i_1}, \dots, A_{1,j,i_{l_{1,j}}}$ , the sets in  $\mathcal{R}$  such that

$$w_1^{-1}.A_{1,j,i_s} \cap C_j \neq \emptyset, \text{ for every } 1 \leq s \leq l_{1,j}.$$

We denote by  $B_{1,1}, \dots, B_{1,k_1}$ , with  $k_1 = \sum_{j=1}^k l_{1,j}$ , the elements of the collection

$$\{w_1^{-1}.A_{1,j,i_s} \cap C_j : 1 \leq s \leq l_{1,j}, 1 \leq j \leq k\}.$$

We have that  $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D, 1 \leq j \leq k_1\}$  is a clopen finite partition of  $X$ . In addition, for every  $1 \leq i \leq k_1$  there exist  $1 \leq j \leq k$  and  $1 \leq s \leq l_{1,j}$  such that  $w_1.B_{1,i} \subseteq A_{1,j,i_s}$ ,  $w_1.B_{1,i} \subseteq w_1.C_j$  and  $\bigcup_{s=1}^{l_j} B_{1,i} = C_j$ . In other words, we have obtained a clopen partition  $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D, 1 \leq j \leq k_1\}$ , satisfying

- For every  $1 \leq j \leq k_1$ , there exist  $A$  in  $\mathcal{R}$  and  $B$  in  $\mathcal{Q}$  such that  $w_1.B_{1,j}$  is contained in  $A \cap B$ .
- $\bigcup_{j=1}^{k_1} B_{1,j} = C$ .

Now, for  $2 \leq n \leq l$ , we suppose that the step  $n - 1$  has produced a finite clopen partition  $\mathcal{Q}_{n-1} = \{w.B_{n-1,j} : w \in D, 1 \leq j \leq k_{n-1}\}$  such that

- For every  $1 \leq j \leq k_{n-1}$  and every  $1 \leq i \leq n - 1$ , there exists  $A$  in  $\mathcal{R}$  and  $B \in \mathcal{Q}$  such that  $w_i.B_{n-1,j}$  is contained in  $A \cap B$ .
- $\bigcup_{j=1}^{k_{n-1}} B_{n-1,j} = C$ .

Step  $n$ : let  $1 \leq j \leq k_{n-1}$ . Consider  $A_{n,j,i_1}, \dots, A_{n,j,i_{l_{n,j}}}$ , the sets in  $\mathcal{R}$  such that

$$w_n^{-1}.A_{n,j,i_s} \cap B_{n-1,j} \neq \emptyset, \text{ for every } 1 \leq s \leq l_{n,j}.$$

We denote by  $B_{n,1}, \dots, B_{n,k_n}$ , with  $k_n = \sum_{j=1}^k l_{n,j}$ , the elements in the collection

$$\{w_n^{-1}.A_{j,i_s} \cap B_{n-1,j} : 1 \leq s \leq l_{n,j}, 1 \leq j \leq k_{n-1}\}.$$

We have  $\mathcal{Q}_n = \{w.B_{n,j} : w \in D, 1 \leq j \leq k_n\}$  is a clopen finite partition of  $X$ . In addition, for every  $1 \leq l \leq k_n$  there exist  $1 \leq j \leq k_{n-1}$  and  $1 \leq s \leq l_{n,j}$  such that  $B_{n,l} \subseteq B_{n-1,j}$  and  $B_{n,l} \subseteq w_n^{-1}.A_{n,j,s}$ . This implies that for every  $1 \leq i \leq n - 1$ ,  $w_i.B_{n,l} \subseteq w_i.B_{n-1,j}$  and by hypothesis,  $w_i.B_{n,l}$  is included in a subset  $A_i$  in  $\mathcal{R}$ . Since  $\bigcup_{l=1}^{k_n} B_{n,l} = \bigcup_{i=1}^{k_{n-1}} B_{n-1,i}$ , the partition  $\mathcal{Q}_n$  satisfies

- For every  $1 \leq j \leq k_n$  and  $1 \leq i \leq n$ , there exist  $A$  in  $\mathcal{R}$  and  $B$  in  $\mathcal{Q}$  such that  $w_i.B_{n,j}$  is contained in  $A \cap B$ .
- $\bigcup_{j=1}^{k_n} B_{n,j} = C$ .

This implies that at the end of the step  $l$ , we obtain a partition

$$\mathcal{R} \wedge \mathcal{Q} = \mathcal{Q}_l = \{w.B_{l,j} : w \in D, 1 \leq j \leq k_l\},$$

which is finer than  $\mathcal{R}$  and  $\mathcal{Q}$ , and which satisfies  $\bigcup_{j=1}^{k_l} B_{n,j} = C$ .

Now we use this algorithm to prove the Proposition. From Corollary 1, there exists a decreasing sequence  $(C_n)_{n \geq 0}$  of clopen subsets of  $X$  and a decreasing sequence  $(\Gamma_n)_{n \geq 0}$  of syndetic subgroups of  $G$  such that  $|\bigcap_{n \geq 0} C_n| = 1$  and  $T_{C_n}(x) = \Gamma_n$  for all  $x \in C_n$ . For every  $n \geq 0$ , we take a subset  $D_n$  of  $G$  containing exactly one representing element in each class of  $G/\Gamma_n$ , and we define

$$\mathcal{Q}_n = \{w.C_n : w \in D_n\}.$$

The collection  $\mathcal{Q}_n$  is a finite clopen partition of  $X$ .

Since  $X$  is a Cantor set, it is always possible to take a sequence  $(\mathcal{R}_n)_{n \geq 0}$  of finite clopen partitions of  $X$  which spans its topology.

We construct the desired sequence  $(\mathcal{P}_n)_{n \geq 0}$  as follows:

- We set  $\mathcal{P}_0 = \mathcal{R}_0 \wedge \mathcal{Q}_0$ .
- For  $n > 0$ . First, we set  $\mathcal{P}'_n = \mathcal{R}_n \wedge \mathcal{Q}_n$ , and then  $\mathcal{P}_n = \mathcal{P}_{n-1} \wedge \mathcal{P}'_n$ .

From this construction we get

$$(\mathcal{P}_n = \{w.C_{n,j} : w \in D_n, 1 \leq j \leq k_n\})_{n \geq 0},$$

a sequence of finite clopen partition of  $X$  satisfying, for every  $n \geq 0$ :

- $\mathcal{P}_n$  is finer than  $\mathcal{P}_{n-1}$  and  $\mathcal{R}_n$ .
- $\bigcup_{j=1}^{k_n} C_{n,j} = C_n$ .

The first point implies  $(\mathcal{P}_n)_{n \geq 0}$  is a nested sequence and that it spans the topology of  $X$ . The second point implies that this sequence verifies conditions 1. and 2. from the Proposition.  $\square$

The group  $G$  is amenable if and only if it has a Følner sequence, that is, a sequence  $(F_n)_{n \geq 0}$  of finite subsets of  $G$  such that for every  $g \in G$

$$\lim_{n \rightarrow \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

**Remark 1.** From [We] we deduce that in the case  $G$  amenable, it is always possible to take  $(D_n)_{n \geq 0}$ , defined as in Proposition 4, as a Følner sequence. Thus in this paper, if  $G$  is amenable, we suppose that  $(D_n)_{n \geq 0}$  is Følner.

Let  $(X, G)$  be a regularly recurrent Cantor system. Consider the sequence of finite clopen partitions of  $X$  as in Proposition 4:

$$(\mathcal{P}_n = \{w.C_{n,k} : w \in D_n, 1 \leq k \leq k_n\})_{n \geq 0}.$$

Let  $n \geq 0$ . The incidence matrix between  $\mathcal{P}_n$  and  $\mathcal{P}_{n+1}$  is  $A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+)$  defined by

$$A_n(i, j) = |\{w \in D_{n+1} : w.C_{n+1,j} \subseteq C_{n,i}\}|.$$

Notice that  $\sum_{i=1}^{k_n} A_n(i, j) = q_{n,j}$  is the number of  $w \in D_{n+1}$  such that  $w.C_{n+1,j} \subseteq C_n$ . Since the set of return times of the points in  $C_n$  to  $C_n$  is equal to  $\Gamma_n$ , the number  $q_{n,j}$  does not depend on  $j$  and it is equal to the number of  $w \in D_{n+1}$  which are in  $\Gamma_n$ . So,  $q_{n,j} = \frac{|D_{n+1}|}{|D_n|}$  for every  $1 \leq j \leq k_{n+1}$ . Consider the set

$$\Delta_n = \{(x_1, \dots, x_{k_n}) \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} x_i = \frac{1}{|D_n|}\}.$$

Since, for every  $1 \leq j \leq k_{n+1}$ ,  $\sum_{i=1}^{k_n} A_n(i, j) = \frac{|D_{n+1}|}{|D_n|}$ , the map  $A_n : \Delta_{n+1} \rightarrow \Delta_n$  is well defined.

Because  $(\mathcal{P}_n)_{n \geq 0}$  is a countable collection of clopen sets that spans the topology of  $X$ , any invariant measure defined on this family of sets extends to a unique invariant measure on the Borel  $\sigma$ -algebra of  $X$ . So, any invariant measure  $\mu$  on  $(\mathcal{P}_n)_{n \geq 0}$  must verify

$$\mu(C_{n,i}) = \sum_{j=1}^{k_{n+1}} A_n(i, j) \mu(C_{n+1,j}), \text{ for every } 1 \leq i \leq k_n \text{ and } n \geq 0,$$

and it is completely determined by this relation. In other words, we can identify an invariant measure with an element in the inverse limit  $\lim_{\leftarrow n} (\Delta_n, A_n)$ . In the next Lemma we provide a sufficient condition to have the reciprocal.

**Lemma 5.** *If  $G$  is amenable then  $\mathcal{M}_G(X)$  is affine-homeomorphic to  $\lim_{\leftarrow n} (\Delta_n, A_n)$ .*

*Proof.* Since  $G$  is amenable, we can suppose that  $(D_n)_{n \geq 0}$  is a Følner sequence (See Remark 1).

Let  $((x_{n,1}, \dots, x_{n,k_n}))_{n \geq 0}$  be an element in  $\lim_{\leftarrow n} (\Delta_n, A_n)$ . It defines a probability measure on  $X$  by setting

$$\mu(u.C_{n,i}) = x_{n,i}, \text{ for every } 1 \leq i \leq k_n, u \in D_n \text{ and } n \geq 0.$$

To show this measure is invariant it is sufficient to show that for every  $n \geq 0$ ,  $1 \leq k \leq k_n$  and  $v \in G$ ,  $\mu(v.C_{n,k}) = \mu(C_{n,k}) = x_{n,k}$ .

Let  $m > n$  and consider the sets

$$J(m, n, k, l) = \{w \in D_m : w.C_{m,l} \subseteq C_{n,k}\}.$$

$$J_1(m, n, k, l) = \{w \in J(m, n, k, l) : vw \in D_m\} \text{ and } J_2(m, n, k, l) = J(m, n, k, l) \setminus J_1(m, n, k, l).$$

We have

$$v.C_{n,k} = \bigcup_{l=1}^{k_m} \bigcup_{w \in J(m,n,k,l)} vw.C_{m,l},$$

and then

$$\mu(v.C_{n,k}) = \sum_{l=1}^{k_m} \sum_{w \in J(m,n,k,l)} \mu(vw.C_{m,l}) = \sum_{l=1}^{k_m} \sum_{w \in J_1(m,n,k,l)} \mu(vw.C_{m,l}) + \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(vw.C_{m,l}).$$

Since  $\mu(u.C_{m,l}) = \mu(C_{m,l})$  for  $u \in D_m$ , we get

$$\begin{aligned} \mu(v.C_{n,k}) &= \sum_{l=1}^{k_m} |J_1(m,n,k,l)| \mu(C_{m,l}) + \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(vw.C_{m,l}) \\ &= \mu(C_{n,k}) - \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(C_{m,l}) + \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(vw.C_{m,l}). \end{aligned}$$

Thus we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(C_{m,l}) + \sum_{l=1}^{k_m} \sum_{w \in J_2(m,n,k,l)} \mu(vw.C_{m,l}).$$

Because  $J_2(m,n,k,l) \subset \{w \in D_m : vw \notin D_m\}$ , we have

$$\begin{aligned} |\mu(v.C_{n,k}) - \mu(C_{n,k})| &\leq \sum_{\{w \in D_m : vw \notin D_m\}} \sum_{l=1}^{k_m} \mu(C_{m,l}) + \sum_{\{w \in D_m : vw \notin D_m\}} \sum_{l=1}^{k_m} \mu(vw.C_{m,l}) \\ &= \sum_{\{w \in D_m : vw \notin D_m\}} \mu\left(\bigcup_{l=1}^{k_m} C_{m,l}\right) + \sum_{\{w \in D_m : vw \notin D_m\}} \mu\left(\bigcup_{l=1}^{k_m} vw.C_{m,l}\right). \end{aligned}$$

Since  $|\{w \in D_m : vw \notin D_m\}| \leq |v.D_m \triangle D_m|$  and  $\mu\left(\bigcup_{l=1}^{k_m} C_{m,l}\right) = \mu\left(\bigcup_{l=1}^{k_m} vw.C_{m,l}\right) = \frac{1}{|D_m|}$ , we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \frac{2|v.D_m \triangle D_m|}{|D_m|}.$$

So, because  $(D_n)_{n \geq 0}$  is Følner, we get  $\mu(v.C_{n,k}) = \mu(C_{n,k})$ .

□

## 5. SEMICOCYCLES

The notion of a semicocycle has been extensively used in the theory of one-dimensional Toeplitz flows (see [Do]). In this section it is not used but we develop it for actions of a residually finite discrete group  $G$  for further utility.

We fix a finite family  $S$  of generators of the group  $G$ , and we suppose furthermore that this family is symmetric ( $S^{-1} = S$ ). We can then associate to the group  $G$  and to the family  $S$  a *Cayley graph*. This graph is defined as follow: the vertices are the elements of  $G$  and two elements  $g_1, g_2$  of the group  $G$  are related by an edge if and only if there exists a element  $s$  of  $S$  such that  $g_2 = s.g_1$ , where  $\cdot$  is the multiplication in the group  $G$ . This graph is endowed with the natural metric of the length path: the distance between two points is the minimal length of paths going from one point to the other, each edge counting for a longer one. This induces on  $G$  a metric  $d$ , which is invariant by the multiplication to the right by any element  $\gamma$  of  $G$ .

Recall that for a residually finite group  $G$  and a decreasing sequence  $(\Gamma_i)_{i \geq 0}$  of syndetic subgroups of  $G$  with  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$ , the stabilizer of  $\mathbf{e} = (e_i)_{i \geq 0}$  in the  $G$ -subodometer  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  is trivial. This defines an immersion  $\tau$  of  $G$  into  $\overleftarrow{G}$ .

**Definition 3.** Let  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  be a  $G$ -subodometer with  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$  and let  $K$  be a compact metric space. A function  $f : G \rightarrow K$  is a *semicycle* on  $\overleftarrow{G}$  if it is continuous with respect to  $\Theta_{\overleftarrow{G}}$ , where  $\Theta_{\overleftarrow{G}}$  is the topology on  $G$  inherited from  $\overleftarrow{G}$  (we identify  $\tau(G)$  with  $G$ ).

The functions  $f : G \rightarrow K$  may be seen as elements of the topological dynamical system  $(K^G, G)$ , where  $K^G$  is endowed with the metrizable product topology and the left-action of  $\gamma \in G$  on  $f = (f(g))_{g \in G} \in K^G$  is the shift action: this means  $\gamma.f = \{f'\}_{g \in G}$ , where  $f'(g) = f(g\gamma)$  for every  $g \in G$ .

The proofs of Theorems 3 and 4 below follow the same ideas as used in [Do] for dimension one.

**Theorem 3.** If  $f \in K^G$  is a semicycle on some subodometer  $\overleftarrow{G}$  then  $f$  is a regularly recurrent point of  $(K^G, G)$ .

*Proof.* Fix  $\epsilon > 0$  and a finite set  $C$  in  $G$ . The pair  $(\epsilon, C)$  determines a basic open set  $V$  in the Tychonov topology. Since  $f$  is continuous on  $G$  for the topology induced by the odometer  $\overleftarrow{G}$ , there exists  $\delta > 0$  such that for every  $g \in C$  and  $g' \in G$ ,  $\text{dist}(g, g') < \delta$  (for the metric inherited from  $\overleftarrow{G}$ ) implies  $d(f(g), f(g')) < \epsilon$  in  $K$ . By definition of a subodometer, there exist a finite index subgroup  $\Gamma$  of  $G$  and a factor map  $\pi : \overleftarrow{G} \rightarrow G/\Gamma$  such that for any element  $w$  of  $G/\Gamma$ ,  $\pi^{-1}(w)$  is a clopen subset of  $\overleftarrow{G}$  with diameter smaller than  $\delta$ . Furthermore, for any  $y \in \pi^{-1}(w)$ ,  $T_{\pi^{-1}(w)}(y)$  is a group conjugated to  $\Gamma$ . Let us consider now the finite index normal subgroup  $H = \bigcap_{g \in G} g\Gamma g^{-1}$ . Since  $\Gamma$  is of finite index in  $G$ , there is just a finite number of groups conjugated to  $\Gamma$  and the former intersection is a finite intersection. The group  $H$  is a subgroup of any group of the kind  $T_{\pi^{-1}(w)}(y)$  with  $w \in G/\Gamma, y \in \pi^{-1}(w)$ . Thus,  $\text{dist}(n'.g, n') < \delta$  for any  $g \in H$  and hence  $d(f(n'.g), f(n')) < \epsilon$  for any  $g \in H$  by the normality of  $H$ . We prove by this way that the  $H$ -orbit of  $f$  is included in  $V$  and then  $f$  is a regularly recurrent point of  $K^G$ .  $\square$

Proposition 2 and Theorem 3 imply that  $(\Omega_G(f), G)$  is a minimal almost 1-1 extension of some subodometer, where  $\Omega_G(f)$  represents the closure orbit of a semicycle  $f$  in  $K^G$  with a trivial stabilizer under the action of  $G$ . Notice that  $\overleftarrow{G}$  need not to be the maximal equicontinuous factor of  $(\Omega_G(f), G)$ , as we will see later.

Let  $f \in K^G$  be a semicycle on a  $G$ -subodometer  $\overleftarrow{G}$ . Since we have identified the group  $G$  with  $G$  embedded in  $\overleftarrow{G}$ , it makes sense to define  $F$  to be the closure of the graph of  $f$  in  $\overleftarrow{G} \times K$  endowed with the product topology,  $F = \overline{\{(g, f(g)) : g \in G\}} \subseteq \overleftarrow{G} \times K$ . Let  $F(\mathbf{g})$  be the set  $\{k \in K : (\mathbf{g}, k) \in F\}$  for  $\mathbf{g} \in \overleftarrow{G}$ .

We call  $C_f$  the set of  $\mathbf{g} \in \overleftarrow{G}$  such that  $|F(\mathbf{g})| = 1$  and  $D_f = \overleftarrow{G} \setminus C_f$ . Since  $f$  is continuous we have that  $F(g) = \{f(g)\}$  for all  $g \in G$ . Thus  $C_f$  is the subset where  $f$  can be continuously extended by  $f(\mathbf{g}) = F(\mathbf{g})$ .

The semicycle  $f$  is said to be *invariant under no rotation* if for every  $\mathbf{h}_1 \neq \mathbf{h}_2 \in \overleftarrow{G}$  there exists a  $g \in G$  such that  $F(g.\mathbf{h}_1) \neq F(g.\mathbf{h}_2)$ .

**Theorem 4.** Let  $(X, G)$  be a minimal topological dynamical system and  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  be a  $G$ -subodometer with  $\bigcap_{i \geq 0} \Gamma_i = \{e\}$ . There exists an almost 1-1 factor  $\pi$  of  $(X, G)$  onto

$(\overleftarrow{G}, G)$  with  $|\pi^{-1}(\mathbf{e})| = 1$  if and only if  $(X, G)$  is conjugated to  $(\Omega_G(f), G)$ , where  $f$  is a semicycle on  $\overleftarrow{G}$ , invariant under no rotation.

*Proof.* Consider the system  $(\Omega_G(f), G)$ . By definition, for every  $x \in \Omega_G(f)$ , there exists a sequence  $(g_i)_i \subset G$  such that for each  $h \in G$ ,  $\lim_i f(hg_i) = x(h)$ . Let  $\mathbf{j} \in \overleftarrow{G}$  be an accumulation point of the sequence  $(g_i)_i$ . We have  $x(h) \in F(h, \mathbf{j})$ . By a straightforward calculation, we check that for each such  $\mathbf{j}$ , the set  $\{(h, \mathbf{j}, x(h)) \mid h \in G\}$  is a dense subset of  $F$ . Since  $f$  is invariant under no rotation,  $\mathbf{j}$  is determined for any  $x$  in a unique way. So we have proved that if  $g_i \cdot f \rightarrow x$  then  $g_i \rightarrow \mathbf{j}$ . The map  $\pi : x \in \Omega_G(f) \mapsto \mathbf{j} \in \overleftarrow{G}$  is a continuous extension onto  $\Omega_G(f)$  of the application  $g \cdot f \mapsto g$ . It is straightforward to check that  $\pi$  is a factor map that sends  $f$  to  $\mathbf{e}$ . If  $\pi(x) = \mathbf{e}$  then  $x(h) \in F(h) = \{f(h)\}$  and  $x(h) = f(h)$  for any  $h \in G$ . Since the system  $(\Omega_G(f), G)$  is minimal,  $\pi$  is an almost 1 to 1 factor map. Conversely, consider a minimal almost 1-1 extension  $(X, G)$  of a  $G$ -subodometer and  $\pi : X \rightarrow \overleftarrow{G}$  the associated factor map. Consider  $x \in X$  such that  $\pi(x)$  has a singleton fiber by  $\pi$  so this is the same for all the elements of its  $G$ -orbit. The map  $f : g \in G \mapsto \pi^{-1}(g \cdot \pi(x)) = g \cdot \pi^{-1}(x) \in X$  is continuous for the induced topology on  $G$ , it is then a semicycle. This is straightforward to check that  $F(\mathbf{j}) = \pi^{-1}(\mathbf{k})$  where  $\mathbf{k} \in \overleftarrow{G}$  is the limit point of the sequence  $(g_i \cdot \pi(x))_i$  with  $(g_i)$  a sequence of  $G$  that converges to  $\mathbf{j}$ . The set  $\pi^{-1}(\mathbf{k})$  does not depend of the choice of the sequence  $(g_i)$ . It is then straightforward to show that  $f$  is invariant under no rotation. The conjugating map from  $(\Omega_G(f), G)$  onto  $(X, G)$  is the projection onto the neutral element coordinate:  $\phi \mapsto \phi(\mathbf{e})$ . By a standard way, we check this application is a homeomorphism which commutes with the  $G$ -action.  $\square$

**Corollary 5.** *A topological dynamical system  $(X, G)$  is a minimal almost 1-1 extension of a free odometer  $(\overleftarrow{G}, G)$  if and only if it is conjugated to  $(\Omega_G(f), G)$ , where  $f$  is a semicycle on  $G$ , invariant under no rotation.*

*Proof.* For a factor map  $p : X \rightarrow \overleftarrow{G}$  and any point  $x \in X$ , by a right multiplication by  $\pi(x)^{-1}$ , we obtain again a factor map that sends the point  $x$  to  $\mathbf{e}$ . The result follows from Theorem 4.  $\square$

## 6. $G$ -TOEPLITZ ARRAYS

In this section, we suppose that  $G$  is a discrete finitely generated group. Let  $\Sigma$  be a finite alphabet and  $\Gamma \subseteq G$  a syndetic subgroup of  $G$ . For  $x = (x(g))_{g \in G} \in \Sigma^G$  we define:

$$Per(x, \Gamma, \sigma) = \{g \in G : x(g\gamma) = \sigma \text{ for all } \gamma \in \Gamma\}, \sigma \in \Sigma,$$

$$Per(x, \Gamma) = \bigcup_{\sigma \in \Sigma} Per(x, \Gamma, \sigma).$$

Clearly for two subgroups  $\Gamma_1$  and  $\Gamma_2$ ,  $\Gamma_1 \subset \Gamma_2$ , we have  $Per(x, \Gamma_2, \sigma) \subset Per(x, \Gamma_1, \sigma)$ . When  $Per(x, \Gamma) \neq \emptyset$  we say that  $\Gamma$  is a *group of periods* of  $x$ . Furthermore,  $Per(x, \Gamma)$  is a subset stable by multiplication to the right by a element of  $\Gamma$ . We say that  $x$  is a  *$G$ -Toeplitz array* (or simply a Toeplitz array) if for all  $g \in G$  there exists  $\Gamma \subseteq G$  syndetic subgroup of  $G$  such that  $g \in Per(x, \Gamma)$ .

**Proposition 5.** *The following statements concerning  $x \in \Sigma^G$  are equivalent:*

- (1)  $x$  is Toeplitz array.
- (2) There exists a sequence of syndetic subgroups  $(\Gamma_n)_{n \geq 0}$ , such that  $\Gamma_{n+1} \subset \Gamma_n$  and  $G = \bigcup_n Per(x, \Gamma_n)$  for all  $n \geq 0$ .
- (3)  $x$  is regularly recurrent.

*Proof.* Let  $D_n$  be the ball of radius  $n$  in  $G$  centered in the neutral element.

Suppose that  $x$  is a Toeplitz array. Since for  $Z_1$  and  $Z_2$ , two groups of period of  $x$ , we have  $Per(x, Z_1) \subset Per(x, Z_1 \cap Z_2)$ , for any  $n \geq 0$ , there exists a syndetic subgroup  $Z_n$  such that  $D_n \subset Per(x, Z_n)$ . Let  $\Gamma_0 = Z_0$  and  $\Gamma_{n+1} = \Gamma_n \cap Z_n$ . The sequence  $(\Gamma_n)_n$  satisfies the statement (2).

Let  $(\Gamma_n)_n$  be a sequence as in statement (2). Let  $C_n$  be the set  $\{y \in \Sigma^G : y(D_n) = x(D_n)\}$  for all  $n \geq 0$ ,  $(C_n)_{n \geq 0}$  is a fundamental system of clopen neighborhoods of  $x$ . Since  $D_n$  is contained in  $Per(x, \Gamma_n)$ , the set of return times of  $x$  to  $C_n$  contains  $\Gamma_n$  which implies that  $x$  is regularly recurrent.

Suppose that  $x$  is regularly recurrent. For  $n \geq 0$  we take  $\Gamma_n$  a syndetic subgroup of  $G$  such that  $\Gamma_n \subseteq T_{C_n}(x)$ . It holds that  $G$  is equal to  $\bigcup_{n \geq 0} Per(x, \Gamma_n)$ , which means that  $x$  is a Toeplitz array.  $\square$

A subshift  $(X, G)$  is a  $G$ -Toeplitz system (or simply a Toeplitz system) if there exists a Toeplitz array  $x$  such that  $X = \Omega_G(x)$ . From Theorem 2 and Proposition 5 we conclude that the family of minimal subshifts which are almost 1-1 extensions of subodometers coincides with the family of Toeplitz systems.

In order to know the maximal equicontinuous factor of a given Toeplitz system, we will introduce the concepts of essential group of periods and period structure.

**Definition 4.** Let  $x \in \Sigma^G$ . A syndetic group  $\Gamma \subset G$  is called an essential group of periods of  $x$  if  $Per(x, \Gamma, \sigma) \subseteq Per(x, g^{-1}\Gamma g, \sigma)g^{-1} = Per(g.x, \Gamma, \sigma)$  for every  $\sigma \in \Sigma$  implies that  $g \in \Gamma$ .

**Lemma 6.** If  $\Gamma$  is an essential group of periods of  $x$  then every group of periods  $\Gamma'$  satisfying  $Per(x, \Gamma) \subseteq Per(x, \Gamma')$  is contained in  $\Gamma$ .

*Proof.* Let  $\Gamma$  be an essential group of periods of  $x$ . Suppose that  $\Gamma'$  is a group of periods such that  $Per(x, \Gamma) \subseteq Per(x, \Gamma')$ . For  $w \in Per(x, \Gamma, \sigma)$  and  $g \in \Gamma'$  we have  $w\gamma g \in Per(x, \Gamma', \sigma)$  for every  $\gamma \in \Gamma$ . This implies that  $x(w\gamma g) = g.x(w\gamma) = \sigma$  for every  $\gamma \in \Gamma$ , which means that  $w \in Per(g.x, \Gamma', \sigma)$ . Because  $\Gamma$  is essential, we conclude that  $g \in \Gamma$  and then  $\Gamma' \subseteq \Gamma$ .  $\square$

**Remark 2.** From Lemma 6 we deduce that the family of the essential groups of periods is contained in the family of the groups generated by essential periods introduced in [Co] for the case  $G = \mathbb{Z}^d$ .

In the following Lemma we show the existence of essential groups of periods.

**Lemma 7.** Let  $x \in \Sigma^G$ . If  $\Gamma \subseteq G$  is a group of periods of  $x$  then there exists  $K \subseteq G$  an essential group of periods of  $x$  such that  $Per(x, \Gamma) \subseteq Per(x, K)$ .

*Proof.* Let  $\Gamma \subseteq G$  be a group of periods of  $x$  and  $\Gamma'$  be a syndetic normal subgroup of  $\Gamma$ . We call  $\hat{\Gamma}'$  the set

$$\bigcup_{g \in G} \{Hg : H \text{ syndetic subgroup of } G \text{ such that } Per(x, \Gamma', \sigma) \subseteq Per(x, g^{-1}Hg, \sigma)g^{-1}, \forall \sigma \in \Sigma\}.$$

Let  $K$  be the group generated by the elements of  $\hat{\Gamma}'$ . Let  $w \in Per(x, \Gamma', \sigma)$ . For any  $\gamma \in \Gamma'$  and any  $Hg \in \hat{\Gamma}'$ , we have  $w\gamma$  belongs to  $Per(x, \Gamma', \sigma) \subseteq Per(x, g^{-1}Hg, \sigma)g^{-1}$ . This implies that for every  $hg \in Hg \in \hat{\Gamma}'$  we have  $w\gamma hg \in Per(x, g^{-1}Hg, \sigma)$ . Since  $\Gamma'$  is a normal subgroup, we get for any  $\gamma \in \Gamma'$  and any  $hg \in Hg \in \hat{\Gamma}'$ ,  $x(whg\gamma) = \sigma$ , which means that  $whg \in Per(x, \Gamma', \sigma)$ . Thus we obtain that for any  $h_1g_1, \dots, h_ng_n$  with  $h_i g_i$  belonging to a set in  $\hat{\Gamma}'$ , we have  $x(h_1g_1 \dots h_ng_n) = \sigma$ . In other words,  $Per(x, \Gamma', \sigma)$  is contained in  $Per(x, K, \sigma)$ . So, we have  $Per(x, \Gamma, \sigma) \subseteq Per(x, \Gamma', \sigma) \subseteq Per(x, K, \sigma)$ . If  $g \in G$  is such that

$Per(x, K, \sigma) \subseteq Per(g.x, K, \sigma) = Per(x, g^{-1}Kg, \sigma)g^{-1}, \forall \sigma \in \Sigma$ , then  $Kg$  belongs to  $\hat{\Gamma}'$ , which implies that  $g$  is in  $K$ .  $\square$

**Corollary 6.** *Let  $x \in \Sigma^G$  be a Toeplitz array. There exists a sequence  $(\Gamma_n)_{n \geq 0}$  of essential group of periods of  $x$  such that  $\Gamma_{n+1} \subseteq \Gamma_n$  and  $\bigcup_{n \geq 0} Per(x, \Gamma_n) = G$ .*

*Proof.* From Proposition 5 (2) we conclude there exists a decreasing sequence  $(\Gamma'_n)_{n \geq 0}$  of syndetic groups of periods of  $x$  such that  $\bigcup_{n \geq 0} Per(x, \Gamma'_n) = G$ . We set  $\Gamma_0$  an essential group of periods of  $x$  such that  $Per(x, \Gamma'_0) \subseteq Per(x, \Gamma_0)$ . For  $n > 0$  we set  $\Gamma''_n = \Gamma'_n \cap \Gamma_{n-1}$  which is a syndetic subgroup of  $G$ , and since  $Per(x, \Gamma_{n-1})$  and  $Per(x, \Gamma'_n)$  are contained in  $Per(x, \Gamma''_n)$ ,  $\Gamma''_n$  is a group of periods of  $x$ . Thus, by Lemma 7, there exists an essential group of periods  $\Gamma_n$ , such that  $Per(x, \Gamma_{n-1}) \subseteq Per(x, \Gamma''_n) \subseteq Per(x, \Gamma_n)$ . Since  $\Gamma_{n-1}$  is an essential group of periods, from Lemma 6 we get  $\Gamma_n \subseteq \Gamma_{n-1}$ . Because  $\bigcup_{n \geq 0} Per(x, \Gamma'_n) = G$ , we deduce  $\bigcup_{n \geq 0} Per(x, \Gamma_n) = G$ .  $\square$

**Definition 5.** *A sequence of groups as in Corollary 6 is called a period structure of  $x$ .*

In the sequel, we will show that from a period structure  $(\Gamma_n)_{n \geq 0}$  of a  $G$ -Toeplitz array  $x$  it is possible to construct a sequence of nested finite clopen partitions of  $\Omega_G(x)$ . From this sequence of partitions it will be easy to define an almost 1-1 factor map between the Toeplitz system  $(\Omega_G(x), G)$  and the odometer  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$ .

Let  $x \in \Sigma^G$  be a Toeplitz array, let  $y \in \Omega_G(x)$  and let  $\Gamma \subseteq G$  be a subgroup of  $G$  with finite index. Since  $(\Omega_\Gamma(y), \Gamma)$  is minimal, if  $\Gamma$  is a group of periods of  $y$  then  $\Omega_\Gamma(y) \subseteq C_\Gamma(y)$ , where

$$C_\Gamma(y) = \{x' \in \Omega_G(x) : Per(x', \Gamma, \sigma) = Per(y, \Gamma, \sigma), \forall \sigma \in \Sigma\}.$$

**Lemma 8.**  $C_\Gamma(y) = \gamma.C_\Gamma(y)$  for every  $\gamma \in \Gamma$ .

*Proof.* Let  $x' \in \gamma.C_\Gamma(y)$ . There exists  $x'' \in C_\Gamma(y)$  such that  $x' = \gamma.x''$ . If  $g \in Per(x', \Gamma, \sigma)$  then  $x'(g\gamma') = \sigma$  for every  $\gamma' \in \Sigma$ . In particular, we have

$$x'(g\gamma'\gamma^{-1}) = \gamma^{-1}.x'(g\gamma') = x''(g\gamma') = \sigma, \forall \gamma' \in \Gamma,$$

which implies  $Per(x', \Gamma, \sigma) \subseteq Per(y, \Gamma, \sigma)$ . On the other hand, if  $g \in Per(x'', \Gamma, \sigma)$  then

$$x''(g\gamma') = x''(g\gamma'\gamma) = \gamma.x''(g\gamma') = x'(g\gamma') = \sigma, \forall \gamma' \in \Gamma,$$

which implies that  $Per(y, \Gamma, \sigma) \subseteq Per(x', \Gamma, \sigma)$ . Thus we obtain that  $\gamma.C_\Gamma(y) \subseteq C_\Gamma(y)$ . Since this is true for every  $\gamma \in \Gamma$ , we conclude that  $\gamma.C_\Gamma(y) = C_\Gamma(y)$ .  $\square$

We will use the following convention: For a  $\Gamma$ -periodic subset  $C$  of  $\Omega_G(x)$ , i.e., such that  $w.C = w'.C$  whenever  $w^{-1}w' \in \Gamma$  we will write  $v.C$  instead of  $w.C$ , where  $v$  is the projection of  $w$  to  $G/\Gamma$ .

**Proposition 6.** *Let  $x \in \Sigma^G$  be a Toeplitz array and let  $y \in \Omega_G(x)$ . If  $\Gamma \subseteq G$  is a subgroup generated by essential periods of  $y$  then  $\Omega_\Gamma(y) = C_\Gamma(y)$  and  $\{w.C_\Gamma(y)\}_{w \in G/\Gamma}$  is a clopen partition of  $\Omega_G(x)$ .*

*Proof.* By Lemma 8,  $C_\Gamma(y)$  is a clopen set and we have  $\Gamma \subseteq T_{C_\Gamma(y)}(x')$  for every  $x' \in C_\Gamma(y)$ . In the sequel, we will show that for  $\Gamma$  a group generated by essential periods, we have  $T_{C_\Gamma(y)}(x') = \Gamma$  for every  $x' \in C_\Gamma(y)$ , which will allow us to conclude.

Suppose that  $g \in \Gamma$  satisfies  $g.y \in C_\Gamma(y)$ . This implies  $Per(g.y, \Gamma, \sigma) = Per(y, \Gamma, \sigma)$  for every  $\sigma \in \Sigma$ . Since  $Per(g.y, \Gamma, \sigma) = Per(y, g^{-1}\Gamma g, \sigma)g^{-1}$ , we obtain  $g \in \Gamma$  because  $\Gamma$  is a group generated by essential periods of  $y$ . By minimality, we conclude that  $T_{C_\Gamma(y)}(x') = \Gamma$  for every  $x' \in C_\Gamma(y)$ . Thus we get that  $\{w.C_\Gamma(y)\}_{w \in G/\Gamma}$  is a collection of disjoint sets. Moreover, this collection is a partition of  $\Omega_G(x)$  because  $w.\Omega_\Gamma(y) \subseteq w.C_\Gamma(y)$  for every  $w \in G/\Gamma$  and  $\{w.\Omega_\Gamma(y)\}_{w \in G/\Gamma}$  is a covering of  $\Omega_G(x)$ . This also implies that  $\Omega_G(x) = C_\Gamma(x)$ .  $\square$



**Proposition 7.** *Let  $x \in \Sigma^G$  be a Toeplitz array. If  $(\Gamma_n)_{n \geq 0}$  is a period structure of  $x$  then the subodometer  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  is the maximal equicontinuous factor of  $(\Omega_G(x), G)$ .*

*Proof.* By Proposition 6, if  $(\Gamma_n)_{n \geq 0}$  is period structure of the Toeplitz array  $x$ , then  $(C_{g.\Gamma_n}(x) : g \in G/\Gamma_n)_{n \geq 0}$  is a sequence of nested clopen partitions of  $\Omega_G(x)$ . This implies that the function  $f_n : \Omega_G(x) \rightarrow G/\Gamma_n$  given by  $f_n(y) = g$  if and only if  $y \in g.C_{\Gamma_n}(x)$  is a well defined continuous function,  $y \in \Omega_G(x)$ ,  $n \geq 0$ . The function  $\pi : \Omega_G(x) \rightarrow \overleftarrow{G}$  given by  $\pi = (f_n)_{n \geq 0}$  is a factor map. Since  $\bigcap_{n \geq 0} C_{\Gamma_n}(x) = \{x\}$ , we have that  $\pi^{-1}\{\mathbf{e}\} = \{x\}$  and then  $\pi$  is an almost 1-1 factor map.  $\square$

**Theorem 5.** *For every subodometer  $\overleftarrow{G}$  there exists a Toeplitz array  $x \in \{0, 1\}^G$  such that  $\overleftarrow{G}$  is the maximal equicontinuous factor of  $(\Omega_G(x), G)$ .*

*Proof.* Let  $\overleftarrow{G} = \lim_{\leftarrow n} (G/\Gamma_n, \pi_n)$  be a subodometer with  $\Gamma_0 = G$ . We distinguish two cases:  
*Case 1:* There exists  $m \geq 0$  such that  $\Gamma_n = \Gamma_m$  for all  $n \geq m$ . In this case  $\overleftarrow{G}$  is the finite group  $G/\Gamma_m$  and then every minimal almost 1-1 extension will be conjugate to  $\overleftarrow{G}$ . For example,  $x \in \{0, 1\}^G$  defined by  $x(v) = 0$  for all  $v \in \Gamma_m$  and  $x(v) = 1$  if not, provides a Toeplitz sequence  $x$  such that  $\overleftarrow{G}$  is the maximal equicontinuous factor of the system associated to  $x$ .

*Case 2:* For every  $m \geq 0$  there exists  $n > m$  such that  $\Gamma_n \neq \Gamma_m$ . In this case we can take a subsequence  $(\Gamma_n)_{n \geq 0}$  such that  $\Gamma_{n+1} \neq \Gamma_n$  and  $[\Gamma_n : \Gamma_{n+1}] \geq 2$  for all  $n \geq 0$ . By Proposition 2,  $\overleftarrow{G}$  is conjugate to the subodometer obtained from this sequence. In order to construct the Toeplitz array  $x$  we will consider a sequence  $(D_n)_{n \geq 0}$  of compact subsets of  $G$  such that:

- for each  $n$ ,  $D_n$  is a fundamental domain of  $\Gamma_n$  (i.e.  $D_n$  contains an unique element of each class of  $G/\Gamma_n$ ). The set  $D_0$  is the singleton set  $\{e\}$ .
- For each  $n$ ,  $D_n \subset D_{n+1}$  and  $D_{n+1} = \bigsqcup_{k \in K_n} D_n.k$  for some finite set  $K_n \subset G$  containing the neutral element  $e$  of  $G$ . By assumption, the cardinal of  $K_n$  is bigger than 2.
- $\bigcup_{n \geq 0} D_n = G$ .

We define now a sequence of subsets of  $G$   $(S_n)_{n \geq 0}$  by induction. Let  $S_0$  be the singleton  $\{e\}$ . Let  $v_1$  be an element of  $D_1$  distinct from  $e$  and let  $S_1 = \{v_1\}$ . For  $n > 1$ , let  $S_n$  be the set  $v_{n-1}.\Gamma_{n-1} \cap D_n \setminus D_{n-1}$  and let  $v_n$  be a point in  $S_n$ . We define then  $x \in \{0, 1\}^G$  by :

$$(2) \quad x(w) = \begin{cases} 0 & \text{if } w \text{ belongs to } \bigcup_{n \geq 0} S_{2n}.\Gamma_{2n+1} \\ 1 & \text{else} \end{cases}$$

Remark that  $x(w) = 1$  for the element  $w$  of  $\bigcup_{n \geq 0} S_{2n+1}.\Gamma_{2n+2}$ . Since  $\bigcup_{j=0}^{n-1} v_j.\Gamma_{j+1} \subseteq \text{Per}(x, \Gamma_n, 0)$  and  $(D_{n-1} \setminus \bigcup_{j=0}^{n-1} v_j.\Gamma_{j+1}) \subseteq \text{Per}(x, \Gamma_n, 1)$ , it holds that  $G = \bigcup_{n \geq 0} \text{Per}(x, \Gamma_n)$  and  $x$  is a Toeplitz array. To conclude that  $\overleftarrow{G}$  is the maximal equicontinuous factor of the system associated to  $x$ , by Proposition 7, it suffices to show that  $(\Gamma_n)_{n \geq 0}$  is a period structure of  $x$ .

Let us prove by induction on  $n$  that  $\Gamma_n$  is a group generated by essential periods of  $x$ . For  $n = 0$ ,  $\Gamma_0 = G$  and this is obviously true. Suppose now that  $n > 0$  and that  $\Gamma_{n-1}$  is a group generated by essential periods. Let  $g \in G$  be such that  $\text{Per}(x, \Gamma_n, \sigma) \subset \text{Per}(x, g^{-1}\Gamma_n g, \sigma).g^{-1}$ , for all  $\sigma$  of  $\{0, 1\}$ . Since  $\Gamma_n \subset \Gamma_{n-1}$ , we have  $\text{Per}(x, \Gamma_{n-1}, \sigma) \subset \text{Per}(x, \Gamma_n, \sigma)$ . Let  $w$  be in  $\text{Per}(x, \Gamma_{n-1}, \sigma)$  and  $\gamma_{n-1}$  in  $\Gamma_{n-1}$ , there exists  $\gamma \in D_n$  and  $\gamma_n \in \Gamma_n$  such that  $\gamma_{n-1} = \gamma\gamma_n$ . Then  $w\gamma_{n-1}\gamma_n^{-1} = w\gamma$  belongs to  $\text{Per}(x, \Gamma_{n-1}, \sigma) \subset \text{Per}(x, g^{-1}\Gamma_n g, \sigma).g^{-1} = \text{Per}(g.x, \Gamma_n, \sigma)$ . So we have  $\sigma = g.x(w\gamma) = g.x(w\gamma.\gamma_n) = g.x(w.\gamma_{n-1})$  and therefor  $w \in \text{Per}(g.x, \Gamma_{n-1}, \sigma) = \text{Per}(x, g^{-1}\Gamma_{n-1}g, \sigma).g^{-1}$  for all  $w \in \text{Per}(x, \Gamma_{n-1}, \sigma)$ . By the hypothesis of induction we get

that  $g$  belongs to  $\Gamma_{n-1}$ .

By the definition of  $x$ , the element  $v_{n-1}$  belongs to  $Per(x, \Gamma_n, \sigma)$  with  $\sigma = x(v_{n-1})$ , so  $x(v_{n-1}.g) = \sigma$ . Since  $g \in \Gamma_{n-1}$  and by the construction of  $x$ ,  $g$  belongs to  $\Gamma_n$  and so  $\Gamma_n$  is a group generated by essential periods of  $x$ .  $\square$

**Remark 3.** *It is interesting to note that when  $\overleftarrow{G}$  is a free odometer, the action of  $G$  on  $\overleftarrow{G}$  is free and minimal. The  $G$ -Toeplitz array  $x$ , constructed as above, is such that  $(\Omega_G(x), G)$  is an extension almost 1-1 of the system  $(\overleftarrow{G}, G)$ , so the action of  $G$  on  $\Omega_G(x)$  is also free and minimal. All the elements of  $\Omega_G(x)$  are not stable by the action of  $G$ . Remark also that, very recently and independently from our work, F. Krieger in [Kr] gives a similar construction of sequence  $G$ -Toeplitz. This kind of examples are, at our knowledge, the only examples given with these properties for a general  $G$ -action.*

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